ENTIRE FUNCTIONS SATISFYING A LINEAR DIFFERENTIAL EQUATION

BY

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ABSTRACT

If f is an entire function and satisfies a certain differential equation, then it is shown that f is of bounded index. This extends a theorem of S. M. Shah.

1. INTRODUCTION

An entire function f is said to be of bounded index if there exists a nonnegative integer N such that

$$\max_{0\leqslant j\leqslant N}\left\{\frac{|f^{(j)}(z)|}{j!}\right\}\geqslant\frac{|f^{(n)}(z)|}{n!}\text{ for all } z\in\mathbb{C} \text{ and all } n=0,1,\ldots,.$$

The least such integer N is called the index of f.

Let $f(z)$ be an entire function satisfying a linear differential equation of the form $\sum_{j=0}^{n} P_j(z)f^{(n-j)}(z)=Q(z)$, where P_j and Q are polynomials and $\deg P_0\geqslant\deg P_j$. Then $f(z)$ is of bounded index [4]. In this paper we show that this theorem holds when $Q(z)$ is replaced by an entire function $g(z)$ of bounded index. We prove

THEOREM 1. *Let $f(z)$ be a transcendental entire function and suppose that $f(z)$ satisfies the differential equation*

$$P_0(z)f^{(n)}(z)+P_1(z)f^{(n-1)}(z)+\ldots+P_n(z)f(z)=g(z), \tag{1}$$

where $P_j(z)$, $j=0,1,\ldots,n$ are polynomials and $P_0(z)$ is of degree not less than that of any $P_j(z)$, and $g(z)$ is an entire function of bounded index. Then f is of bounded index.

An entire function F is said to be of bounded value distribution in the sense of TURAN [3, problem 2.28] if there exists an integer p such that the equation $F(z)=w$ has at most p roots in any disc of radius 1.

HAYMAN [3] and FRICKE [1] showed that an entire function F is of bounded value distribution if and only if F' is of bounded index. Thus we have the following consequence.

COROLLARY 2. *If $F(z)$ is an entire function such that $F'(z)=f(z)$ satisfies the differential equation* (1) *then $F(z)$ is of bounded value distribution.*

2. Proof

For the proof of Theorem 1 we need the following result due to Hayman [3].

Theorem A. *An entire function f is of bounded index if and only if there exist an integer $N \geqslant 0$ and a constant $C > 0$ such that*

$$|f^{(N+1)}(z)| \leqslant C \max_{0 \leqslant j \leqslant N} \{|f^{(j)}(z)|\} \text{ for all } z \in \mathbb{C}.$$

Proof of Theorem 1. If $g(z)$ is of bounded index then by Theorem A there exist $N \geqslant 0$ and $C > 0$ such that

$$|g^{(N+1)}(z)| \leqslant C \max_{0 \leqslant j \leqslant N} \{|g^{(j)}(z)|\} \text{ for all } z \in \mathbb{C}.$$

Thus,

$$\begin{aligned}
|g^{(N+1)}(z)| &= \left| \sum_{j=0}^{n} \frac{d^{N+1}}{dz^{N+1}} P_j(z) f^{(n-j)}(z) \right| \\
&= \left| \sum_{j=0}^{n} \sum_{k=0}^{N+1} \binom{N+1}{k} P_j^{(k)}(z) f^{(n-j+N+1-k)}(z) \right| \\
&\leqslant C \max_{0 \leqslant j \leqslant N} \{|g^{(j)}(z)|\} \\
&= C \max_{0 \leqslant j \leqslant N} \left\{ \left| \sum_{k=0}^{n} \sum_{t=0}^{j} \binom{j}{t} P_k^{(t)}(z) f^{(n-k+j-t)}(z) \right| \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|P_0(z) f^{(n+N+1)}(z)| \leqslant{} & \left| \sum_{j=1}^{n} \sum_{k=0}^{N+1} \binom{N+1}{k} P_j^{(k)}(z) f^{(n-j+N+1-k)}(z) \right. \\
& \left. + \sum_{k=1}^{N+1} \binom{N+1}{k} P_0^{(k)}(z) f^{(n+N+1-k)}(z) \right| \\
& + C \max_{0 \leqslant j \leqslant N} \left\{ \left| \sum_{k=0}^{n} \sum_{t=0}^{j} \binom{j}{t} P_k^{(t)}(z) f^{(n-k+j-t)}(z) \right| \right\} \\
\leqslant{} & (C+1)(n+1)(N+2)(N+1)! \max_{\substack{0 \leqslant j \leqslant n \\ 0 \leqslant k \leqslant N+1}} \{|P_j^{(k)}(z)|\} \times \\
& \max_{0 \leqslant j \leqslant N+n} \{|f^{(j)}(z)|\}.
\end{aligned}$$

Now, since $\deg P_0 > \deg P_j$, $j = 0, 1, \ldots, n$, there exist $M > 0$ and $R > 0$ such that

$$M|P_0(z)| \geqslant |P_j^{(k)}(z)| \text{ for all } j = 0, 1, \ldots, n \text{ and } k = 0, 1, \ldots, N+1$$
$$\text{and for all } |z| \geqslant R.$$

Thus,

$$|f^{(n+N+1)}(z)| \leqslant (C+1)(n+1)(N+2)(N+1)! \; M \max_{0 \leqslant j \leqslant N+n} \{|f^{(j)}(z)|\} \text{ for } |z| \geqslant R.$$

Hence, for $S = (C+1)(n+1)(N+2)(N+1)!\ M$ and $T = n+N$,

$$|f^{(T+1)}(z)| \leqslant S \max_{0 \leqslant j \leqslant T} \{|f^{(j)}(z)|\} \text{ for } |z| \geqslant R.$$

This inequality holds for $|z| < R$ [4; p. 133] if we replace S by a suitable constant. Therefore, by Theorem A, f is of bounded index. q.e.d.

REMARK: Theorem 1 can also be proven using similar methods as in [4]. The authors also want to thank J. P. Beauchamp for his helpful suggestion in the proof of Theorem 1.

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